Path integral in the simplest Regge calculus model

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Abstract

The simplest (3+1)D Regge calculus model (with three-dimensional discrete space and continuous time) is considered which describes evolution of the simplest closed two-tetrahedron piecewise flat manifold in the continuous time. The measure in the path integral which describes canonical quantisation of the model in terms of area bivectors and connections as independent variables is found. It is shown that selfdual-antiselfdual splitting of the variables simplifies the integral although does not admit complete separation of (anti-)selfdual sector.

Path integral is probably the most efficient way to quantisation of the discrete theory. In particular, this approach is widely used in Regge calculus, including the physical 4D case [1, 2]. The path integral measure is usually chosen as the simplest among the invariant ones. Another approach is based on constructing state sum invariants of the 4D Regge manifolds [3, 4, 5] in analogy with how it is done in the 3D case where such invariants being functionals of only 2D triangulated boundary of such manifold exist and do not depend on the refinement of the interior tetrahedrons [6, 7, 8, 9, 10]. Thus, quantum measure is defined there by topological requirements.

If description of the system is known in the canonical form, the most natural choice of path integral measure is that based on the canonical quantisation. Knowing canonical form of the theory requires that action for the system in the continuous time be known. In the given paper we consider path integral just for the simplest 3D Regge manifold evolving in the continuous time. Canonical form of this model as limiting case of the completely discrete 4D Regge manifold has been considered in [11] in the area bivector and connection variables on the base of the canonical formulation of the general Regge manifold [12]. The 3D Regge manifold consists of two similar tetrahedrons with mutually identified vertices. Remarkable feature of the related continuous time Regge calculus action is it's notational analogy and correspondence of constraints with the case of the continuum general relativity (GR) action in the Hilbert-Palatini (HP) form, whereas the number of the degrees of freedom is finite, and "large distances" (i.e. those considerably larger than the size of tetrahedron) are absent. Therefore the path integral for this system can be considered as that modelling path integral for the continuum GR on condition that long-distance effects (gravity waves) are excluded. The typical feature of the path integral measure obtained is that it vanishes for symmetrical configurations of the system. This indicates that quantum fluctuations at these points are definitive.

Also path integral is convenient to handle the problem of implementing the so-called Ashtekar variables [13] in the Regge calculus (see review paper [14]) on quantum level. From general viewpoint, there are the two possible approaches to this problem: to put Ashtekar theory on the Regge lattice or to formulate Regge calculus in the Ashtekar-like variables. The recent approaches [14] seem to be of the first type, we try in our work the second one via splitting the antisymmetric tensor variables into the selfdual and antiselfdual parts. It turns out that although there is no complete separation of selfdual and antiselfdual sectors as it occurs in the continuum GR, the considered procedure simplifies path integral considerably.

We start with the canonical structure of our model [11]. The Lagrangian with all the constraints added reads:

$$\begin{aligned} L = \sum_{\mu} \pi^{\mu} \circ \bar{\Omega}_{\mu} \dot{\Omega}_{\mu} \quad &+ \quad \sum_{\mu > \nu} \dot{\psi}_{\mu\nu} \pi^{\mu} \circ \pi^{\nu} - \sum_{\mu} v_{\mu} H_{\mu} - h \circ \sum_{\mu} \pi^{\mu} - \tilde{h} \circ \sum_{\mu} \Omega_{\mu} \pi^{\mu} \bar{\Omega}_{\mu} \\ &+ \quad \sum_{\mu > \nu} \psi_{\mu\nu} \Delta(\pi^{\mu} \circ \pi^{\nu}) + \sum_{\mu} \kappa_{\mu} \pi^{\mu} * \pi^{\mu} + \sum_{\mu > \nu} \kappa_{\mu\nu} \pi^{\mu} * \pi^{\nu}. \end{aligned} \tag{1}$$

Here μ, ν, λ, ρ,... $= 0$, 1, 2, 3 run over four vertices of tetrahedrons; $\pi^{\mu}_{ab} = -\pi^{\mu}_{ba}$ are bivectors of the triangles opposite (in 3D leaf) to the vertices μ; a, b, c, d,... $= 0$, 1,

2, 3 are local frame SO(3,1) indices (SO(4) in the case of Euclidean signature); $\Omega^{ab}_\mu \subset$ SO(3,1); $A \circ B \equiv (1/2)\mathrm{tr} A\bar{B}$ for the bivectors A^{ab} and B^{ab}; ${}^*\!A_{ab} \equiv (1/2)\varepsilon_{abcd}A^{cd}$; $A * B \equiv A \circ ({}^*\!B)$; h^{ab}, $\tilde{h}^{ab}$ are multipliers at Gaussian constraints; the scalars v_μ enter also $\Delta \equiv \sum_\mu v_\mu \Delta_\mu$ and are, therefore, multipliers at the Hamiltonian constraints which are not H_μ but $\mathcal{H}_\mu \equiv H_\mu$ - $\Delta_\mu \sum_{\nu>\lambda} \psi_{\nu\lambda}\pi^\nu \circ \pi^\lambda$ and

$$
\begin{aligned}
\Delta_\mu \pi^\nu &\equiv \sum_{\lambda,\rho} \varepsilon_{\mu\nu\lambda\rho} n_{\mu\lambda}, && (2)\\
H_\mu &= \sum_\nu |n_{\mu\nu}| \arcsin \frac{n_{\mu\nu} \circ R_{\mu\nu}}{|n_{\mu\nu}|} + \varphi^\nu_\mu \pi^\nu \circ \Delta_\mu \pi^\nu, && (3)\\
n_{\mu\nu} &= \sum_{\lambda,\rho} \varepsilon_{\mu\nu\lambda\rho} \left(u^\lambda_\mu \pi^\rho + \frac{1}{2}[\pi^\lambda, \pi^\rho] \right), && (4)\\
R_{\mu\nu} &= \frac{1}{2} \sum_{\lambda,\rho \neq \mu,\nu} \left(\bar{\Omega}_{\mu\rho} \Omega_{\mu\lambda} \right)^{\varepsilon_{\mu\nu\lambda\rho}}, && (5)\\
\Omega_{\mu\nu} &= \Omega_\nu \Pi^\nu_\mu, && (6)\\
\Pi^\nu_\mu &\equiv \exp(\varphi^\nu_\mu \pi^\nu + {}^*\varphi^\nu_\mu {}^*\pi^\nu). && (7)
\end{aligned}
$$

The completely antisymmetric symbol is defined so that $\varepsilon_{0123} = +1$. The u^ν_μ, φ^ν_μ, ${}^*\varphi^\nu_\mu$ are nondynamical independent variables (not appearing in the kinetic term). These enter Lagrangian nonlinearly. The $\psi_{\mu\nu}$, κ_μ and $\kappa_{\mu\nu}$ are variables entering L linearly of which $\psi_{\mu\nu}$ are dynamical ones. Of the six variables $\psi_{\mu\nu}$ (or $\kappa_{\mu\nu}$) only two are independent ones, and in [11] the ψ_{12}, ψ_{23} (and κ_{12}, κ_{23}) were chosen as the only nonzero ones. Now, in order to formulate the theory in the form explicitly symmetrical w.r.t. the permutation of the vertices 0, 1, 2, 3 we assume that all $\psi_{\mu\nu}$, $\kappa_{\mu\nu}$ are not identically zero, but are subject to the following four relations on $\psi_{\mu\nu}$:

$$\Psi_\mu \equiv \sum_{\nu,\lambda \neq \mu} \psi_{\nu\lambda} = 0, \tag{8}$$

and four ones on $\kappa_{\mu\nu}$:

$$\sum_{\nu,\lambda \neq \mu} \kappa_{\nu\lambda} = 0. \tag{9}$$

Let us take into account these relations on $\kappa_{\mu\nu}$ with the help of Lagrange multipliers C_μ. Then we can consider $\kappa_{\mu\nu}$ as six independent variables, coefficients at the following constraints:

$$Z_{\mu\nu} \equiv \pi^\mu * \pi^\nu + \sum_{\lambda \neq \mu,\nu} C_\lambda = 0. \tag{10}$$

The C_μ are considered then as independent variables.

Now passing to the Hamiltonian formalism we need the extended phase space coordinatised by the following canonical pairs (p, q): (π^μ, Ω_μ), $(\tilde{\psi}^{\mu\nu}, \psi_{\mu\nu})$, $(\tilde{C}^\mu, C_\mu)$. The full system of constraints Φ consists of the first class η and second class Θ ones. The coefficients at η in L are h, $\tilde{h}$ and κ_μ. The Θ consists of $\mathcal{H}_\mu$, Ψ_μ, $Z_{\mu\nu}$ and the following constraints on the newly defined momenta:

$$\tilde{C}^\mu = 0 \tag{11}$$

and

$$\tilde{\Psi}^{\mu\nu} \equiv \tilde{\psi}^{\mu\nu} - \pi^\mu \circ \pi^\nu = 0. \tag{12}$$

Next we introduce the gauge conditions $\chi = 0$ by the number of the first class constraints η such that $\mathrm{Det}\{\eta, \chi\} \neq 0$. Then the path integral measure takes the standard form:

$$\begin{aligned} d\mu_\chi &= e^{i\int Ldt}\delta(\Phi_\chi)\mathrm{Det}^{1/2}\{\Phi_\chi, \Phi_\chi\}DpDq \\ &= e^{i\int Ldt}\mathrm{Det}^{1/2}\{\Theta, \Theta\}\delta(\Theta)\delta(\eta)\delta(\chi)\mathrm{Det}\{\eta, \chi\}DpDq \end{aligned} \tag{13}$$

where $\Phi_\chi = (\Phi, \chi)$. For the specific form of the kinetic term for $(p, q) = (\pi, \Omega)$ one gets $DpDq = d^6\pi\mathcal{D}^6\Omega$, $\mathcal{D}^6\Omega$ being the Haar measure on SO(3,1), and under the Poisson brackets (PB) one should imply the expression

$$\{f, g\} = \pi \circ [f_\pi, g_\pi] + f_\pi \circ \bar{\Omega} g_\Omega - g_\pi \circ \bar{\Omega} f_\Omega \tag{14}$$

(for the functions of purely π, Ω; taking into account dependence on other variables is straightforward). In [11, 12] this form was used as that describing Hamiltonian dynamics. Although quite expectable, this can be proved by considering standard kinetic term $(1/2)\mathrm{tr}\bar{P}\dot{\Omega}$ in the further extended phase space of canonical pairs (P, Ω), $(\tilde{\psi}, \psi)$, $(\tilde{C}, C)$ and substituting into (13) instead of Φ_χ the set $\Phi^\prime_\chi = (\Phi_\chi, \tau)$ extended by inclusion the constraints τ specifying the 4×4 matrices P, Ω to have the form $\Omega \in$ SO(3,1), $P = \Omega\pi$, $\bar{\pi} = -\pi$. Then $\mathrm{Det}\{\Phi^\prime_\chi, \Phi^\prime_\chi\} = \mathrm{Det}\{\Phi_\chi, \Phi_\chi\}_{\mathrm{D}(\tau)}$ where $\{\cdot, \cdot\}_{\mathrm{D}(\tau)}$ are Dirac brackets (DB) w.r.t. the set τ. The $\{\cdot, \cdot\}_{\mathrm{D}(\tau)}$ turn out to coincide just with the earlier defined brackets (14). The Haar measure just arises when integrating over $DPD\Omega$ the newly arising δ-functions of τ,

$$\delta(\tau) = \prod_\mu \delta(\bar{\Omega}_\mu \Omega_\mu - 1)\delta(\bar{\Omega}_\mu P^\mu + \bar{P}^\mu \Omega_\mu). \tag{15}$$

Now consider an important consequence of finiteness (in the Euclidean case) of the volume of the gauge group. The gauge transformations are some rotations changing Ω_μ, ${}^*\varphi^\mu_\nu$. At the same time, the gauge fixing factor $\delta(\chi)\mathrm{Det}\{\eta, \chi\}$ in the path integral measure just arises upon dividing by the volume of the group of gauge transformations (generated by I class constraints η via PB $\{\eta, \cdot\}$). Therefore, if we consider Euclidean case (or Lorentzian one in the sense of analytical continuation from the Euclidean case) we can insert this volume back into measure simply omitting gauge fixing factor without a risk of getting a new infinity. The measure takes the form

$$d\mu = e^{i\int Ldt}\mathrm{Det}^{1/2}\{\Theta, \Theta\}\delta(\Theta)\delta(\eta)DpDq. \tag{16}$$

It is convenient to divide the set of second class constraints Θ into two second class subsets ϑ and ι so that $\mathrm{Det}\{\Theta, \Theta\} = \mathrm{Det}\{\vartheta, \vartheta\}_{\mathrm{D}(\iota)}$. Let the set ϑ consists of the four Hamiltonian constraints $\mathcal{H}_\mu$, the ι be the set of other second class constraints Ψ_μ, $Z_{\mu\nu}$,

$\tilde{C}^\mu$, $\tilde{\Psi}^{\mu\nu}$. To calculate the DB $\{\mathcal{H}_\mu, \mathcal{H}_\nu\}_{\mathrm{D}(\iota)}$ let us first define the projection $\mathcal{H}^\perp_\mu$ of $\mathcal{H}_\mu$ onto the subspace orthogonal w.r.t. the PB to the space of differentials of the set ι:

$$\mathcal{H}^\perp_\mu = \mathcal{H}_\mu - \sum_{\lambda>\nu} A^{\lambda\nu}_\mu Z_{\lambda\nu} - \sum_\nu B_{\mu\nu}\tilde{C}^\nu - \sum_\nu E^\nu_\mu \Psi_\mu - \sum_{\lambda>\nu} F_{\mu\lambda\nu}\tilde{\Psi}^{\lambda\nu} \tag{17}$$

where the coefficients A, B, E, F are defined so that $\{\mathcal{H}^\perp_\mu, \iota\} = 0$. For example, commuting $\mathcal{H}^\perp_\mu$ with constraints $Z_{\lambda\nu}$ and Ψ_λ gives, respectively,

$$-\sum_{\rho\neq\lambda,\nu} B_{\mu\rho} + \left\{\sum_{\rho>\epsilon} F_{\mu\rho\epsilon}\pi^\rho \circ \pi^\epsilon, \pi^\lambda * \pi^\nu\right\} + \{\mathcal{H}_\mu, \pi^\lambda * \pi^\nu\} = 0 \tag{18}$$

and

$$\sum_{\nu,\rho\neq\lambda} F_{\mu\nu\rho} = 0. \tag{19}$$

These eqs. allow to find $B_{\mu\nu}$ and $F_{\mu\nu\lambda}$. Commutation with $\tilde{C}^\nu$ and $\tilde{\Psi}^{\lambda\nu}$ allows to find $A^{\lambda\nu}_\mu$ and E^ν_μ from the analogous system. In the calculations of such kind commutators between different π-bilinears, $\pi^\mu \circ \pi^\nu$ and $\pi^\mu * \pi^\nu$, arise. These commutators by (14) reduce to the following trilinears:

$$\begin{aligned}\pi^\mu \circ [\pi^\nu, \pi^\lambda] &= \pi^\mu * [\pi^\nu, {}^*\pi^\lambda] = 0,\\ \pi^\mu * [\pi^\nu, \pi^\lambda] &= -V^2 \sum_\rho \varepsilon_{\rho\mu\nu\lambda},\end{aligned} \tag{20}$$

V being the volume of the parallelepiped spanned by the links issuing from any tetrahedron vertex. These equalities easily follow from explicit expressions of bivectors in terms of link vectors. The consequence is that commutators of such bilinears of the type of scalar-scalar and pseudoscalar-pseudoscalar vanish, only those of the scalar-pseudoscalar type are nonzero. For example,

$$\left\{\sum_{\nu>\lambda} F_{\mu\nu\lambda}\pi^\nu \circ \pi^\lambda, \pi^0 * \pi^1\right\} = V^2(F_{12} + F_{03} - F_{31} - F_{02}). \tag{21}$$

Finding A, B, E, F we can find the DB of interest,

$$\begin{aligned}\{\mathcal{H}_\mu, \mathcal{H}_\nu\}_{\mathrm{D}(\iota)} &\equiv \{\mathcal{H}^\perp_\mu, \mathcal{H}^\perp_\nu\} = \{H_\mu, H_\nu\} + \sum_{\lambda>\rho} \psi_{\lambda\rho}[\Delta_\nu, \Delta_\mu]\pi^\lambda \circ \pi^\rho\\ &- \sum_{\substack{\lambda>\rho\\ \sigma>\phi}} \{H_\nu, \pi^\lambda * \pi^\rho\} K_{\lambda\rho\sigma\phi}(\Delta^\Omega_\mu - \Delta_\mu)(\pi^\sigma \circ \pi^\phi)\\ &+ \sum_{\substack{\lambda>\rho\\ \sigma>\phi}} \{H_\mu, \pi^\lambda * \pi^\rho\} K_{\lambda\rho\sigma\phi}(\Delta^\Omega_\nu - \Delta_\nu)(\pi^\sigma \circ \pi^\phi)\end{aligned} \tag{22}$$

where we have introduced the notation $\Delta^\Omega_\mu(\cdot) \equiv \{H_\mu, \cdot\}$. The reason for such notation is that for the particular case of the solution to the eqs. of motion for Ω when $R_{\mu\nu} n_{\mu\nu} \bar{R}_{\mu\nu}$

$= n_{\mu\nu}$ for each $\mu\nu$ (i.e. when Ω is a true rotation between the local frames) the operator Δ^{Ω}_{μ} coincides with Δ_{μ} (variation at shifting the point μ to the next time leaf). The $K_{\mu\nu\lambda\rho}$ can be called an inversed to the matrix $\{\pi^{\mu} \circ \pi^{\nu}, \pi^{\lambda} * \pi^{\rho}\}$,

$$K_{\mu\nu\lambda\rho} = \frac{1}{12V^2} \sum_{\epsilon} (\varepsilon_{\lambda\rho\nu\epsilon}\delta^{\rho}_{\mu} + \varepsilon_{\lambda\rho\mu\epsilon}\delta^{\rho}_{\nu} - \varepsilon_{\lambda\rho\nu\epsilon}\delta^{\lambda}_{\mu} - \varepsilon_{\lambda\rho\mu\epsilon}\delta^{\lambda}_{\nu}). \tag{23}$$

(The matrix $\{\pi^{\mu} \circ \pi^{\nu}, \pi^{\lambda} * \pi^{\rho}\}$ is degenerate, and a strict sense to this term is given by solving the eqs. (18), (19).) These DB can be further transformed using the following two facts. First, note that the operation

$$f \rightarrow g = f - \sum_{\substack{\mu > \nu \\ \lambda > \rho}} \{f, \pi^{\mu} * \pi^{\nu}\} K_{\mu\nu\lambda\rho} \pi^{\lambda} \circ \pi^{\rho} \tag{24}$$

on the space of linear combinations of scalars $\pi^{\mu} \circ \pi^{\nu}$ projects onto the subspace of such combinations of only the squares $|\pi^{\mu}|^2$. On these squares $\Delta^{\Omega}_{\mu} = \Delta_{\mu}$ (as consequence of the eqs. of motion for φ^{μ}_{ν}). This allows to simplify contribution to $\{H_{\mu}, g\}$ which results from the dependence of f on the scalars $\pi^{\mu} \circ \pi^{\nu}$. Second, there is some complication connected with the (exponential) dependence of $\Omega_{\mu\nu}$ on π^{ν} (6). Suppressing indices, consider a function $f(\pi, \tilde{\Omega})$ of π and $\tilde{\Omega} = \Omega\Pi$, $\Pi = \exp(\varphi\pi + {}^{*}\varphi\, {}^{*}\pi)$. The following formula can be obtained:

$$\left(\frac{\partial f}{\partial \pi}\right)_{\Omega} = \left(\frac{\partial f}{\partial \pi}\right)_{\tilde{\Omega}} - \frac{1}{|\pi|^2}\left[\pi, \Pi\bar{\tilde{\Omega}}\frac{\partial f}{\partial \tilde{\Omega}}\bar{\Pi} - \bar{\tilde{\Omega}}\frac{\partial f}{\partial \tilde{\Omega}}\right], \tag{25}$$

where both derivatives $\partial/\partial\pi$ and $\bar{\tilde{\Omega}}\partial/\partial\tilde{\Omega}$ are implied to be antisymmetrised and subscript at $\partial f/\partial \pi$ denotes the quantity considered as constant at the differentiating. This allows to simplify expression for the PB of the two functions $f(\pi, \Omega_1)$ and $g(\pi, \Omega_2)$ where $\Omega_i = \Omega\Pi_i$, $\Pi_i = \exp(\varphi_i\pi + {}^{*}\varphi_i\, {}^{*}\pi)$:

$$\{f, g\} = \{f, g\}_{\Pi} + \frac{1}{|\pi|^2}\pi \circ \left(\left[\bar{\Omega}_1\frac{\partial f}{\partial \Omega_1}, \bar{\Omega}_2\frac{\partial g}{\partial \Omega_2}\right] - \left[\Pi_1\bar{\Omega}_1\frac{\partial f}{\partial \Omega_1}\bar{\Pi}_1, \Pi_2\bar{\Omega}_2\frac{\partial g}{\partial \Omega_2}\bar{\Pi}_2\right]\right). \tag{26}$$

Here $\{f, g\}_{\Pi}$ are "naive" PB taken in the formal assumption that Π_i are constants. The second term in this formula is responsible for the dependence of Π_i on π. It vanishes at $\Pi_1 = \Pi_2$ as it should because exponential can be eliminated by redefining Ω in this case.

To represent the DB of the Hamiltonian constraints of interest in the form resembling that of commutators in the continuum GR (in the HP form) we introduce the notation

$$h_{\mu} = \sum_{\nu} \frac{n_{\mu\nu} \circ R_{\mu\nu}}{\cos \alpha_{\mu\nu}} \tag{27}$$

where $\alpha_{\mu\nu}$ is angle defect, $\sin \alpha_{\mu\nu} = n_{\mu\nu} \circ R_{\mu\nu}/|n_{\mu\nu}|$. The h_{μ} arises when we differentiate $\mathcal{H}_{\mu}$ as complex function over phase variables on which it depends through the angle defects. Note also that the dependence of $\mathcal{H}_{\mu}$ on phase variables through the nondynamical

ones u^ν_μ, φ^ν_μ, ${}^*\!\varphi^\nu_\mu$ need not be taken into account at the calculating the first derivatives due to the eqs. of motion for the nondynamical variables. The DB of interest read

$$\begin{aligned}
\{\mathcal{H}_\mu, \mathcal{H}_\nu\}_{\mathrm{D}(\iota)} &= \{h_\mu, h_\nu\}_{\alpha,\Pi} + \sum_\lambda \frac{1}{|\pi^\lambda|^2}\pi^\lambda \circ \left(\left[\bar{\Omega}_{\lambda\mu}\frac{\partial h_\mu}{\partial \Omega_{\lambda\mu}}, \bar{\Omega}_{\lambda\nu}\frac{\partial h_\nu}{\partial \Omega_{\lambda\nu}} \right] \right. \\
&- \left. \left[\Pi^\lambda_\mu \bar{\Omega}_{\lambda\mu}\frac{\partial h_\mu}{\partial \Omega_{\lambda\mu}}\bar{\Pi}^\lambda_\mu, \Pi^\lambda_\nu \bar{\Omega}_{\lambda\nu}\frac{\partial h_\nu}{\partial \Omega_{\lambda\nu}}\bar{\Pi}^\lambda_\nu \right] \right) \\
&+ \sum_\lambda [(\alpha_{\nu\lambda} - \tan\alpha_{\nu\lambda})\Delta_\mu |n_{\nu\lambda}| - (\alpha_{\mu\lambda} - \tan\alpha_{\mu\lambda})\Delta_\nu |n_{\mu\lambda}|] \\
&+ \frac{1}{2}\sum_\lambda (\varphi^\lambda_\nu \Delta_\mu \Delta_\nu - \varphi^\lambda_\mu \Delta_\nu \Delta_\mu)(\pi^\lambda \circ \pi^\lambda) - \sum_{\lambda > \rho} \psi_{\lambda\rho}[\Delta_\mu, \Delta_\nu](\pi^\lambda \circ \pi^\rho) \\
&+ \sum_{\substack{\lambda>\rho \\ \sigma>\phi}} \{h_\mu, \pi^\lambda * \pi^\rho\}_{\alpha,\Pi} K_{\lambda\rho\sigma\phi}[\{h_\nu, \pi^\sigma \circ \pi^\phi\}_{\alpha,\Pi} - \Delta_\nu(\pi^\sigma \circ \pi^\phi)] \\
&- \sum_{\substack{\lambda>\rho \\ \sigma>\phi}} \{h_\nu, \pi^\lambda * \pi^\rho\}_{\alpha,\Pi} K_{\lambda\rho\sigma\phi}[\{h_\mu, \pi^\sigma \circ \pi^\phi\}_{\alpha,\Pi} - \Delta_\mu(\pi^\sigma \circ \pi^\phi)] \qquad (28)
\end{aligned}$$

where $\Pi^\nu_\mu = \exp(\varphi^\nu_\mu \pi^\nu + {}^*\!\varphi^\nu_\mu {}^*\!\pi^\nu)$, and the subscript α at the PB means that defect angles in the denominator in the definition of h_μ (27) are considered formally as constants. The first term in the formula obtained is notationally close analog of the PB of the combinations of vector (with coefficients u^ν_μ) and Hamiltonian constraints in the continuum GR.

The DB obtained define then the Jacobian factor entering path integral:

$$\begin{aligned}
\mathrm{Det}^{1/2}\{\Theta, \Theta\} &= |\{\mathcal{H}_0, \mathcal{H}_1\}_{\mathrm{D}(\iota)}\{\mathcal{H}_2, \mathcal{H}_3\}_{\mathrm{D}(\iota)} + \{\mathcal{H}_0, \mathcal{H}_2\}_{\mathrm{D}(\iota)}\{\mathcal{H}_3, \mathcal{H}_1\}_{\mathrm{D}(\iota)} \\
&+ \{\mathcal{H}_0, \mathcal{H}_3\}_{\mathrm{D}(\iota)}\{\mathcal{H}_1, \mathcal{H}_2\}_{\mathrm{D}(\iota)}|. \qquad (29)
\end{aligned}$$

The rest factors follow by integrating δ-functions in $d\mu$. Integration over $d^4\tilde{C}$, $d^6\tilde{\psi}$ and $d^6\pi^0$ eliminates $\delta^4(\tilde{C})\,\delta^6(\tilde{\Psi})\,\delta^6(\sum_\mu \pi^\mu)$. Integration over d^4C reduces $\delta^6(Z)$ to δ's of some two independent constraints of the type of $\pi^\mu * \pi^\nu$, $\mu \neq \nu$. Together with $\delta^4(\{\pi^\mu * \pi^\mu\})$ this gives $\delta^6(\{\pi^\alpha * \pi^\beta\})$ (α, β, $\gamma = 1, 2, 3$). Finally, the $\delta^4(\psi)$ annihilate four integrations in $d^6\psi$, and we are left with, e.g., $d^2\psi = d\psi_{21}\, d\psi_{32}$ while $\psi_{10} = \psi_{32}$, $\psi_{20} = \psi_{31} = -\psi_{21} - \psi_{32}$, $\psi_{30} = \psi_{21}$. The result reads

$$\begin{aligned}
d\mu &= \exp\left\{ i \int dt \left[\sum_\alpha \pi^\alpha \circ (\bar{\Omega}_\alpha \dot{\Omega}_\alpha - \bar{\Omega}_0 \dot{\Omega}_0) + \sum_{\lambda>\nu} \dot{\psi}_{\lambda\nu} \pi^\lambda \circ \pi^\nu \right] \right\} \mathrm{Det}^{1/2}\{\Theta, \Theta\} \\
&\quad \delta^6 \left[\sum_\alpha (\Omega_\alpha \pi^\alpha \bar{\Omega}_\alpha - \Omega_0 \pi^\alpha \bar{\Omega}_0) \right] \prod_\mu \delta \left[H_\mu - \sum_{\lambda>\nu} \psi_{\lambda\nu} \Delta_\mu (\pi^\lambda \circ \pi^\nu) \right] \\
&\quad \prod_{\alpha \geq \beta} \delta(\pi^\alpha * \pi^\beta) d^2\psi \prod_\alpha d^6\pi^\alpha \prod_\mu \mathcal{D}^6 \Omega_\mu. \qquad (30)
\end{aligned}$$

Some two of the four $\delta(\mathcal{H}_\mu)$'s can, in principle, be integrated over $d^2\psi$ and the variables $\psi_{\mu\nu}$ be excluded as functions of π, Ω. Note that despite of that ψ seem to enter $\mathcal{H}_\mu$ linearly, the implicit dependence on ψ through u, φ, ${}^*\!\varphi$ is much more complex.

Due to the Jacobian factor the measure vanishes for symmetrical configurations of the system for which the DB $\{\mathcal{H}_\mu, \mathcal{H}_\nu\}_{\mathrm{D}(\iota)}$ vanish.

The remaining δ-functions of $\pi^\mu * \pi^\nu$ reflect the tetrad structure of bivectors, in particular, possibility to place the tetrahedron into 3D space and consider it's area vectors instead of bivectors. To pass to the 3D vector formulation in the explicitly invariant way one can choose splitting the bivectors and generators of the connections into selfdual and antiselfdual parts. To this end we expand

$$\pi^\mu_{ab} = {}^+\pi^\mu_{ab} + {}^-\pi^\mu_{ab}, \quad {}^\pm\pi^\mu_{ab} = {}^\pm\Sigma^i_{ab} \, {}^\pm\pi^\mu_i/2 \tag{31}$$

where ${}^\pm\Sigma^i_{ab}$ form basis of (anti-) selfdual matrices obeying algebra of Pauli ones times $\sqrt{-1}$. The ${}^\pm\vec{\pi}^\mu = \{ {}^\pm\pi^\mu_i | i = 1, 2, 3\}$ are vectors in an abstract complex 3D space. The generator ω_μ of the rotation $\Omega = \exp \omega_\mu$ can be decomposed in the same way into ${}^+\omega_\mu$ and ${}^-\omega_\mu$ which act on the abstract space via vector product $- {}^\pm\vec{\omega}_\mu \times (\cdot)$. Denote by ${}^\pm O_\mu$ $= \exp(- {}^\pm\vec{\omega}_\mu \times (\cdot))$ thus obtained representation of ${}^\pm\Omega_\mu = \exp({}^\pm\omega_\mu)$ in the abstract space. The constraints $\pi^\alpha * \pi^\beta = 0$ look as

$${}^-\vec{\pi}^\alpha \cdot {}^-\vec{\pi}^\beta = {}^+\vec{\pi}^\alpha \cdot {}^+\vec{\pi}^\beta. \tag{32}$$

This defines ${}^-\vec{\pi}^\alpha$ in terms of ${}^+\vec{\pi}^\alpha$ up to an overall SO(3,C) rotation U so that

$${}^-\vec{\pi}^\alpha = U\vec{\pi}^\alpha, \quad \vec{\pi}^\alpha \equiv {}^+\vec{\pi}^\alpha. \tag{33}$$

Substituting this into 30 we find that U can be absorbed by ${}^-O_\mu$ so let us denote

$$G_\mu \equiv {}^-O_\mu U, \quad O_\mu \equiv {}^+O_\mu. \tag{34}$$

The Lagrangian on the constraint surface (kinetic term) takes the form

$$L = \sum_\alpha \frac{1}{4} \varepsilon^{ikl} \pi^\alpha_i (\bar{O}_\alpha \dot{O}_\alpha - \bar{O}_0 \dot{O}_0 + \bar{G}_\alpha \dot{G}_\alpha - \bar{G}_0 \dot{G}_0)_{kl} + \sum_{\mu > \nu} \dot{\psi}_{\mu\nu} \vec{\pi}^\mu \cdot \vec{\pi}^\nu. \tag{35}$$

The Gaussian constraint splits into

$$\begin{aligned} \sum_\alpha (O_\alpha - O_0)\vec{\pi}^\alpha &= 0, & (36)\\ \sum_\alpha (G_\alpha - G_0)\vec{\pi}^\alpha &= 0. & (37) \end{aligned}$$

Analogously, for curvature matrix $R_{\mu\nu}$ denote by ${}^\pm P_{\mu\nu}$ representations of ${}^\pm R_{\mu\nu}$ ($R_{\mu\nu} = {}^+R_{\mu\nu} \; {}^-R_{\mu\nu} = {}^-R_{\mu\nu} \; {}^+R_{\mu\nu}$) in the abstract space. Consider for definiteness $R_{01} = \bar{\Omega}_{02} \, \Omega_{03}$. Absorbing U by ${}^-P_{01}$ we find

$$\begin{aligned} P_{01} &\equiv {}^+P_{01} = \exp({}^+\varphi^2_0 \vec{\pi}^2 \times (\cdot)) \bar{O}_2 O_3 \exp(- {}^+\varphi^3_0 \vec{\pi}^3 \times (\cdot)), & (38)\\ Q_{01} &\equiv \bar{U} \, {}^-P_{01} U = \exp({}^-\varphi^2_0 \vec{\pi}^2 \times (\cdot)) \bar{G}_2 G_3 \exp(- {}^-\varphi^3_0 \vec{\pi}^3 \times (\cdot)), & (39)\\ {}^\pm\varphi^\nu_\mu &\equiv \varphi^\nu_\mu \pm {}^*\varphi^\nu_\mu. & (40) \end{aligned}$$

The corresponding contribution to the Hamiltonian reads

$$|n_{01}|\alpha_{01} = |\vec{n}_{01}| \arcsin \frac{n^i_{01}\varepsilon_{ikl}}{8|\vec{n}_{01}|}(P^{kl}_{01}\sqrt{1+\mathrm{tr}Q_{01}} + Q^{kl}_{01}\sqrt{1+\mathrm{tr}P_{01}}) \tag{41}$$

where $\vec{n}_{01} = u^2_0\vec{\pi}^3$ - $u^3_0\vec{\pi}^2$ - $\vec{\pi}^2 \times \vec{\pi}^3$.

Return now to the path integral. Integration of six δ-functions $\delta(\vec{\pi}^\alpha \cdot \vec{\pi}^\beta - {}^-\vec{\pi}^\alpha \cdot {}^-\vec{\pi}^\beta)$ over $d^3\, {}^-\vec{\pi}^1\ d^3\, {}^-\vec{\pi}^2\ d^3\, {}^-\vec{\pi}^3$ results in $V^{-2}\mathcal{D}^3 U$ ($V^2 = \vec{\pi}^1 \times \vec{\pi}^2 \cdot \vec{\pi}^3$). On the other hand, since U is absorbed by "-" components of connection matrices and due to the invariance of the measure $\mathcal{D}^6\Omega_\mu = \mathcal{D}^3\, {}^+O_\mu \mathcal{D}^3\, {}^-O_\mu = \mathcal{D}^3 O_\mu \mathcal{D}^3 G_\mu$ the integral over $\mathcal{D}^3 U$ decouples giving the volume of the gauge subgroup. This leads to the following replacement in the measure (30):

$$\prod_{\alpha\geq\beta}\delta(\pi^\alpha * \pi^\beta)\prod_\mu \mathcal{D}^6\Omega_\mu \prod_\alpha d^6\pi^\alpha \Rightarrow \frac{1}{\vec{\pi}^1 \times \vec{\pi}^2 \cdot \vec{\pi}^3}\prod_\alpha d^3\vec{\pi}^\alpha \prod_\mu \mathcal{D}^3 O_\mu \mathcal{D}^3 G_\mu \tag{42}$$

The procedure of the derivation of the Ashtekar action by the selfdual-antiselfdual splitting of the continuum GR action in the HP form is known in detail (see, e.g., reviews [15, 16]). We find the following difference of our discrete case from the continuum one: now there is no the constraints like the so-called "reality conditions" of the continuum GR which would relate G_μ and O_μ. Rather we have a theory with one set of area vectors $\vec{\pi}^\mu$ but two sets of connections G_μ and O_μ. The situation is complicated by the fact that the dependences of the Hamiltonian on G_μ and O_μ does not split, see (41). Nevertheless, this procedure remains useful, for it allows to remove a part of the gauge degrees of freedom in an invariant way.

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